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1990 J. Phys. A: Math. Gen. 23 3945

(http://iopscience.iop.org/0305-4470/23/17/024)

J. Phys. A: Math. Gen. 23 (1990) 3945–3952. Printed in the UK

Weyl-invariant random Hamiltonians and their relation to translational-invariant random potentials on Landau levels

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Received 13 March 1990

Abstract. We construct and discuss classes of random Hamiltonians on the infinite-dimensional Hilbert space of a quantum system with a Euclidean configuration space. By construction, the corresponding probability distributions are invariant under the action of the Weyl group. For particular classes of Weyl-invariant random Hamiltonians we establish a relation to translational-invariant random potentials restricted to the Hilbert space of a single Landau level, that is, an eigenspace of the standard Hamiltonian for a charged particle confined to the plane and subjected to a perpendicular constant magnetic field.

1. Introduction

Physical systems, whose properties are unknown in detail, are often successfully described by models possessing both fixed as well as randomly chosen properties.

Prominent models are provided by random matrices caricaturing a complicated Hamiltonian on one of its finite-dimensional invariant subspaces of the physical Hilbert space. The probability distributions of these random matrices are typically chosen to be invariant under transformations which conserve the most important physical symmetries. Initially, they have been studied to understand the spectra of highly excited heavy nuclei (Mehta 1967, Brody *et al* 1981). More recent applications are in the field of mesoscopic electronic systems (Brody *et al* 1981) and quantum chaos (Seligman and Nishioka 1986, Saitô and Aizawa 1989).

Other models, often studied in the context of disordered electronic systems, are standard Hamiltonians with a random potential acting as an operator on an infinite-dimensional Hilbert space (Lifshitz *et al* 1988, Kirsch 1989). A topic of recent interest is the random Landau model, that is, the standard Hamiltonian of a charged particle constrained to the plane and under the influence of a perpendicular constant magnetic field and a random potential. It is believed that this model is adequate to describe the localization phenomena in two-dimensional electronic structures (Landwehr 1987, 1989). A simplification of the model, valid for high magnetic fields, is often made by restricting the Hamiltonian to the still infinite-dimensional subspace of a single Landau level.

The aim of the present paper is twofold. First, we construct and discuss classes of random Hamiltonians on the infinite-dimensional Hilbert space of a quantum system with a Euclidean configuration space. By construction, the corresponding probability

0305-4470/90/173945+08$03.50

distributions are invariant under the action of the Weyl group (Barut and Raczka 1980). Second, for particular classes of Weyl-invariant random Hamiltonians we establish a relation to translational-invariant random potentials on single Landau levels.

In order to illustrate the relation, we infer from known results (Wegner 1983, Brèzin *et al* 1984, Klein and Perez 1985) the averaged spectral function for those Weyl-invariant random Hamiltonians which are related to delta-correlated random or white-noise potentials on the lowest Landau level.

2. Weyl-invariant random Hamiltonians

To begin with, we consider a quantum mechanical system whose configuration space is the real line $\mathbb{R}$. As usual the system's observables corresponding to the position and the momentum are represented by the Hermitian operators Q and P acting on the Hilbert space $L^2(\mathbb{R})$ of square-integrable complex-valued functions on $\mathbb{R}$. They obey the canonical commutation relation

$$(\mathrm{i}/\hbar)\,(PQ - QP) = \mathbf{1}. \tag{2.1}$$

Throughout the paper we make use of a set of coherent-state vectors (Klauder and Skagerstam 1985, Perelomov 1986)

$$|\Phi(p,q)\rangle := \exp\left[\mathrm{i}(pQ - qP)/\hbar\right]|\Phi\rangle \tag{2.2}$$

associated with a normalized reference vector $|\Phi\rangle$ and labelled by points (p,q) of the classical phase space

$$|\Phi\rangle \in L^2(\mathbb{R}) \qquad \langle\Phi|\Phi\rangle = 1 \qquad (p,q) \in \mathbb{R}^2. \tag{2.3}$$

These states resolve the identity operator in the sense

$$\int \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi\hbar}\; |\Phi(p,q)\rangle\,\langle\Phi(p,q)| = \mathbf{1}. \tag{2.4}$$

For a fixed $|\Phi\rangle$, we choose to quantize a classical Hamiltonian $h : \mathbb{R}^2 \to \mathbb{R}$ to the Hamiltonian

$$H := \int \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi\hbar}\; h(p,q)\,|\Phi(p,q)\rangle\,\langle\Phi(p,q)| \tag{2.5}$$

of the quantum system acting as an Hermitian operator on $L^2(\mathbb{R})$.

When h is not a prescribed but a random field, equation (2.5) defines a corresponding random Hamiltonian H. In the following we will only consider homogeneous random fields (Gel'fand and Vilenkin 1964). For these fields the probability distribution of the shifted field $h(p-p', q-q')$ equals the one of $h(p,q)$. Denoting the associated average by an overbar, homogeneity is equivalent to the fact that the characteristic functional

$$C(j) := \overline{\exp\left(-\mathrm{i}\int \mathrm{d}p\mathrm{d}q\; j(p,q)\,h(p,q)\right)} \tag{2.6}$$

does not change its value, when the function $j(p,q)$ is replaced by $j(p+p',q+q')$. The homogeneity of the random field h implies the Weyl invariance of the corresponding random Hamiltonian H, meaning that the Weyl-transformed random Hamiltonian $\exp\{\mathrm{i}(p'Q-q'P)/\hbar\}\, H \exp\{-\mathrm{i}(p'Q-q'P)/\hbar\}$ has the same distribution as H. Equivalently, the characteristic functional

$$\overline{\exp\left[-\mathrm{i}\,\mathrm{Tr}\,(JH)\right]} = C(j_\Phi) \qquad j_\Phi(p,q) := \langle\Phi(p,q)|\, J\, |\Phi(p,q)\rangle \tag{2.7}$$

does not change its value, when the operator J on $L^2(\mathbb{R})$ is replaced by $\exp\{-\mathrm{i}(p'Q-q'P)/\hbar\}\, J \exp\{\mathrm{i}(p'Q-q'P)/\hbar\}$. The Weyl invariance follows from the commutation relation (2.1) in the form

$$\begin{aligned}\exp[\mathrm{i}(p'Q-q'P)/\hbar]\, &\exp[\mathrm{i}(pQ-qP)/\hbar] \\ &= \exp[\mathrm{i}(p'q-pq')/2\hbar]\, \exp[\mathrm{i}((p+p')Q-(q+q')P)/\hbar].\end{aligned} \tag{2.8}$$

Weyl-invariant random Hamiltonians are different from random standard Hamiltonians $\frac{1}{2}P^2+u(Q)$ arising from (spatially) translational-invariant classical random potentials $u(q)$. While random standard Hamiltonians are often used in the theory of disordered electronic systems (Lifshitz *et al* 1988, Kirsch 1989), see also the next section, Weyl-invariant random Hamiltonians are similar to the unitary-invariant random matrices (Mehta 1967, Brody *et al* 1981) considered for modelling heavy nuclei and quantum chaos (Seligman and Nishioka 1986, Saitô and Aizawa 1989).

For Weyl-invariant random Hamiltonians one in particular obtains

$$\exp[\mathrm{i}(pQ-qP)/\hbar]\, \overline{f(H)}\, \exp[-\mathrm{i}(pQ-qP)/\hbar] = \overline{f(H)} \tag{2.9}$$

for a wide class of complex-valued functions $f:\mathbb{R}\to\mathbb{C}$. Because of the irreducibility of P and Q on $L^2(\mathbb{R})$, equation (2.9) implies that $\overline{f(H)}$ is proportional to the identity operator. The proportionality factor in general depends on the reference vector $|\Phi\rangle$ and the probability distribution of the random field h, for example

$$\begin{aligned}\overline{H} &= \overline{h(0,0)}\,\mathbf{1} \\ \overline{H^2} &= \int \frac{\mathrm{d}p\mathrm{d}q}{2\pi\hbar}\, \overline{h(0,0)\,h(p,q)}\, \left|\langle\Phi|\Phi(p,q)\rangle\right|^2\, \mathbf{1}.\end{aligned} \tag{2.10}$$

However, for simple random fields, as the constantly correlated random fields

$$C(j) = \int \mathrm{d}\varepsilon\, \overline{\delta(\varepsilon - h(0,0))}\, \exp\left(-\mathrm{i}\varepsilon \int \mathrm{d}p\mathrm{d}q\, j(p,q)\right) \tag{2.11}$$

and the Cauchy–Lorentz white-noise field

$$C(j) = \exp\left(-\gamma \int \mathrm{d}p\mathrm{d}q\, |j(p,q)|\right) \qquad \gamma > 0 \tag{2.12}$$

$\overline{f(H)}$ does not depend on $|\Phi\rangle$. In these cases it is given by

$$\overline{f(H)} = \int \mathrm{d}\varepsilon\, f(\varepsilon)\, \overline{\delta(\varepsilon - h(0,0))}\, \mathbf{1} \tag{2.13}$$

and depends only on the one-point probability density $\overline{\delta(\varepsilon - h(0,0))}$. While in the case (2.11) this probability density is arbitrary, for (2.12) it reads

$$\overline{\delta(\varepsilon - h(0,0))} = \frac{\gamma/\pi}{\varepsilon^2 + \gamma^2}. \tag{2.14}$$

For a proof of (2.13) we refer to the Chernoff-type representation (Berezin 1972, 1975, Exner 1985, Klauder and Skagerstam 1985)

$$\exp\{-\mathrm{i}tH/\hbar\} = \lim_{N\to\infty} \left(\int \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi\hbar} \exp\{-\mathrm{i}th(p,q)/N\hbar\} \, |\Phi(p,q)\rangle \, \langle\Phi(p,q)| \right)^N \tag{2.15}$$

and to an argumentation analogous to one given by Broderix *et al* (1987).

For more general random fields h and general reference vectors $|\Phi\rangle$ we have no explicit results, at least not for general functions f.

In the following we will gain further insight for the particular case, where $|\Phi\rangle$ is chosen as the nth eigenvector $|\Omega_n\rangle$ of a one-dimensional harmonic oscillator with frequency $\omega > 0$ and unit mass

$$\tfrac{1}{2}\left(P^2 + \omega^2 Q^2\right) = \hbar\omega \sum_{n=0}^{\infty} \left(n + \tfrac{1}{2}\right) |\Omega_n\rangle \, \langle\Omega_n| \tag{2.16}$$

$$\langle\Omega_n|\Omega_{n'}\rangle = \delta_{n,n'} \qquad \sum_{n=0}^{\infty} |\Omega_n\rangle \, \langle\Omega_n| = \mathbf{1}. \tag{2.17}$$

By this choice the scalar product of the coherent-state vectors (2.2) is explicitly given as

$$\begin{aligned}\langle\Omega_n(p,q)|\Omega_n(p',q')\rangle &= \exp\left\{\mathrm{i}(p'q - pq')/2\hbar - \left[(p-p')^2 + \omega^2(q-q')^2\right]/4\hbar\omega\right\} \\ &\quad \times \mathrm{L}_n\left(\left[(p-p')^2 + \omega^2(q-q')^2\right]/2\hbar\omega\right).\end{aligned} \tag{2.18}$$

Here

$$\mathrm{L}_n(\xi) := \frac{1}{n!}\, \mathrm{e}^{\xi} \frac{\mathrm{d}^n}{\mathrm{d}\xi^n} \left(\mathrm{e}^{-\xi}\, \xi^n\right) \tag{2.19}$$

is the Laguerre polynomial of nth order.

We denote the Hamiltonian (2.5) corresponding to h and $|\Phi\rangle = |\Omega_n\rangle$ by H_n. For $n = 0$ the underlying quantization rule is nothing else but antinormal ordering (Cahill and Glauber 1969, Berezin 1971).

3. Relation to translational-invariant random potentials on Landau levels

In this section a connection between the above defined random Hamiltonians H_n and random potentials on the subspace of the nth Landau level is established.

The operators of momentum and position on the Hilbert space $L^2(\mathbb{R}^2)$ of square-integrable functions on the plane $\mathbb{R}^2$ are symbolized by (P_1, P_2) and (Q_1, Q_2), respectively.

In the symmetric gauge the Hamiltonian K for a charged particle with unit mass in the plane subjected to a perpendicular constant magnetic field is given by

$$K = \tfrac{1}{2}\left(P_1 + \tfrac{1}{2}\omega Q_2\right)^2 + \tfrac{1}{2}\left(P_2 - \tfrac{1}{2}\omega Q_1\right)^2 . \tag{3.1}$$

Here $\omega > 0$ denotes the cyclotron frequency of the system. We write the spectral resolution of K as

$$K = \hbar\omega \sum_{n=0}^{\infty} \left(n + \tfrac{1}{2}\right) E_n \tag{3.2}$$

where E_n is the projection operator on the subspace $E_n L^2(\mathbb{R}^2)$ of the nth Landau level $(n + \frac{1}{2})\hbar\omega$. One has

$$E_n^\dagger E_{n'} = \delta_{n,n'} E_n \qquad \sum_{n=0}^{\infty} E_n = \mathbf{1}. \tag{3.3}$$

The position representation of E_n reads

$$\langle x_1, x_2 | \, E_n \, | x_1', x_2' \rangle = \frac{\omega}{2\pi\hbar} \langle \Omega_n(\omega x_1, x_2) | \Omega_n(\omega x_1', x_2') \rangle \tag{3.4}$$

compare with equation (2.18).

In contrast to (2.5), we now use the one-dimensional classical Hamiltonian h to construct the Hermitian multiplication operator

$$V := \int \mathrm{d}x_1 \mathrm{d}x_2 \; v(x_1, x_2) \, | x_1, x_2 \rangle \, \langle x_1, x_2 | \tag{3.5}$$

on $L^2(\mathbb{R}^2)$ where the function

$$v(x_1, x_2) := h(\omega x_1, x_2) \tag{3.6}$$

for fixed frequency ω, is interpreted as a classical potential for the charged particle in the plane and V as its quantum counterpart.

Besides this potential V we consider the Hamiltonian H_n defined in (2.5) with respect to $|\Omega_n\rangle \langle \Omega_n|$. Then the following identity

$$\langle x_1, x_2 | \, E_n f(E_n V E_n) E_n \, | x_1', x_2' \rangle = \frac{\omega}{2\pi\hbar} \langle \Omega_n(\omega x_1, x_2) | \, f(H_n) \, | \Omega_n(\omega x_1', x_2') \rangle \tag{3.7}$$

holds for a wide class of functions $f : \mathbb{R} \to \mathbb{C}$. This is a consequence of (3.3) and (3.4). For $n = 0$ the above identity has been used by Daubechies and Klauder (1986) in their definition of Feynman path-integrals by means of Wiener regularization.

If h is a random field with a given distribution, the distributions of the induced random Hamiltonian H_n and the random potential V are fully determined. If, in addition, the distribution of h is chosen so that the distribution of v is independent of ω, the Hamiltonian $K + V$ on $L^2(\mathbb{R}^2)$ characterizes the random Landau model. For high magnetic fields it seems reasonable to neglect a possible mixing between Landau levels, that is, to investigate $\sum_{n=0}^{\infty} E_n(K + V)E_n$ instead of $K + V$. To study the contribution of $E_n(K + V)E_n$ on the subspace $E_n L^2(\mathbb{R}^2)$ of the nth Landau level, the identity (3.7) can be applied. Often, for very high magnetic fields the model is even restricted to the subspace $E_0 L^2(\mathbb{R}^2)$ of the lowest Landau level.

Summarizing, homogeneous random fields h generate both Weyl-invariant random Hamiltonians H_n on $L^2(\mathbb{R})$ and translational-invariant random potentials V on $L^2(\mathbb{R}^2)$ which are related through (3.7). In particular, the corresponding averaged spectral functions may be written as

$$\overline{\delta(\varepsilon - H_n)} = \rho_n(\varepsilon)\, \mathbf{1} \tag{3.8}$$

$$\overline{E_n \delta(\varepsilon - E_n V E_n) E_n} = \rho_n(\varepsilon)\, E_n \tag{3.9}$$

with the same probability density ρ_n on the real line $\mathbb{R}$. The first two moments of ρ_n can be obtained from (2.10) in combination with (2.18). The proportionality (3.8) is due to the remark below (2.9). To our knowledge the proportionality (3.9) was first shown for general n and general translational-invariant random potentials by Klein and Perez (1985). Here it follows from applying (3.8) to (3.7) after averaging. As a consequence of (3.9) and (3.4) the averaged density of states per area of the random Landau model, restricted to the subspace of the nth Landau level, is seen to be

$$\langle x_1, x_2 |\, \overline{E_n \delta(\varepsilon - E_n(K+V)E_n)E_n}\, | x_1, x_2 \rangle = \frac{\omega}{2\pi\hbar}\, \rho_n\big(\varepsilon - \big(n + \tfrac{1}{2}\big)\hbar\omega\big)\,. \tag{3.10}$$

It is of considerable interest that ρ_0 can explicitly be calculated for general homogeneous white-noise fields. These fields are defined by characteristic functionals of the form

$$C(j) = \exp\left(\int \mathrm{d}p\, \mathrm{d}q\; g(j(p,q))\right) \tag{3.11}$$

where $a \mapsto \exp\{\mu\hbar g(a)\}$ has to be the Fourier transform of a probability distribution on the real line for all $\mu > 0$ (Gel'fand and Vilenkin 1969).

The result implied by (3.11) reads

$$\rho_0(\varepsilon) = \frac{1}{\pi} \mathrm{Im} \left\{ \frac{\partial}{\partial \varepsilon} \ln \left[\int_0^\infty \mathrm{d}\tau\, \exp\left(2\pi \mathrm{i} \varepsilon \tau + 2\pi\hbar \int_0^\tau \frac{\mathrm{d}b}{b} g(b/\hbar) \right) \right] \right\} \tag{3.12}$$

as has been shown by Wegner (1983) and Brézin *et al* (1984) in the context of the random Landau model; see also (Klein and Perez 1985) for a non-perturbative derivation. Due to the absence of a length scale in (3.11) the spectral probability density is independent of the frequency ω, provided g is independent of ω.

As it should be, (3.12) is consistent with (2.13) for the Cauchy–Lorentz white-noise field, $g(a) = -\gamma|a|$. Another example, being most similar to the prominent Gaussian random matrices (Mehta 1967, Brody *et al* 1981), is the Gaussian white-noise field, that is, $g(a) = -\nu^2 a^2/2$. In this case ρ_0 exhibits Gaussian tails for $|\varepsilon| \to \infty$. For a picture see (Brézin *et al* 1984).

4. Additional remarks

We close with four remarks.

(i) It would be interesting to find spectral quantities more general than those in (3.8) and (3.9), for example $\overline{\delta(\varepsilon - H_n)\, A\, \delta(\varepsilon' - H_n)}$ with a non-random operator A.

(ii) The generalization of the statements in section 2 to systems with the d-dimensional configuration space $\mathbb{R}^d$ is merely a matter of notation.

(iii) The relation (3.7) of Weyl-invariant random Hamiltonians to translational-invariant random potentials on Landau levels can be generalized to $d \geq 2$ by considering d non-interacting particles in the plane.

(iv) Extensions to non-flat configuration spaces require non-trivial modifications, possibly along some lines in Berezin (1975) and Klauder and Onofri (1989).

Note added. Recently we have succeeded in showing that the spectral function exhibits Gaussian tails for general homogeneous Gaussian random fields

$$\lim_{\varepsilon\to\pm\infty} \frac{1}{\varepsilon^2} \ln \overline{\delta(\varepsilon - H)} = -\frac{1}{2\Gamma^2}.$$

The constant is given by the variational problem

$$\Gamma^2 = \sup_{\substack{|\Psi\rangle \in L^2(\mathbb{R}) \\ \langle\Psi|\Psi\rangle = 1}} \overline{\left(\langle\Psi|\left(H - \overline{H}\right)|\Psi\rangle\right)^2} \leq \langle\Phi|\, \overline{\left(H - \overline{H}\right)^2}\, |\Phi\rangle .$$

In a forthcoming publication we will supply a proof and determine Γ^2 explicitly for $|\Phi\rangle = |\Omega_n\rangle$ and the Gaussian covariance

$$\overline{h(p,q)\, h(0,0)} - \overline{h(0,0)}^2 = \sigma^2 \exp\left[-\left(p^2 + \omega^2 q^2\right)/2\lambda^2\omega^2\right]$$

to

$$\Gamma_n^2 = \sigma^2 \left(\frac{\lambda^2\omega}{\lambda^2\omega + 2\hbar}\right)^{n+1} \mathrm{P}_n\left(\frac{\left(\lambda^2\omega + \hbar\right)^2 + \hbar^2}{(\lambda^2\omega + \hbar)^2 - \hbar^2}\right)$$

where

$$\mathrm{P}_n(\xi) := \frac{1}{n!2^n} \frac{\mathrm{d}^n}{\mathrm{d}\xi^n} \left(\xi^2 - 1\right)^n$$

denotes the nth Legendre polynomial.

These relations generalize results conjectured earlier in the context of the random Landau model (Apel 1987, Benedict 1987).

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